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Tracking error with minimum guarantee constraints

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Abstract. In recent years the popularity of indexing has greatly increased in financial markets and many different families of products have been introduced. Often these products also have a minimum guarantee in the form of a minimum rate of return at specified dates or a minimum level of wealth at the end of the horizon. Period of declining stock market returns together with low interest rate levels on Treasury bonds make it more difficult to meet these liabilities. We formulate a dynamic asset allocation problem which takes into account the conflicting objectives of a minimum guaranteed return and of an upside capture of the risky asset returns. To combine these goals we formulate a double tracking error problem using asymmetric tracking error measures in the multistage stochastic programming framework.

Keywords: Minimum guarantee, benchmark, tracking error, dynamic asset allocation, scenario.

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1 Introduction

The simultaneous presence of a benchmark and a minimum guaranteed return characterizes many structured financial products. The objective is to attract potential investors who express an interest in high stock market returns but also are not risk-seeking enough to fully accept the volatility of this investment and require a cushion. This problem is of interest also for the asset allocation choices for pension funds both in the case of defined benefits (which can be linked to the return of the funds) and defined contribution schemes in order to be able to attract members to the fund. Moreover many life insurance products include an option on a minimum guaranteed return and a minimum amount can be guaranteed by a fund manager for credibility reasons. Thus the proper choice of an asset allocation model is of interest not only for investment funds or insurance companies which offer products with investment components but also for pension fund industry.

In the literature there are contributions which discuss the two components separately, there are contributions which discuss the tracking error problem when a VaR, CVaR or Maximum Drawdown (MD) constraint is introduced mainly in static framework, but very few contributions address the dynamic portfolio management problem when both a minimum guarantee and a tracking error objectives are present, see for example [14]. To jointly model these goals we work in the stochastic programming framework since it proved to be flexible enough to deal with many different issues which arise in the formulation and solution of these problems. We do not consider the point of view of an investor who wants to maximize the expected utility of his wealth along the planning horizon or at the end of the investment period. Instead we consider the point of view of a manager of a fund, thus representing a collection of investors, who is responsible for the management of a portfolio connected with financial products which offer not only a minimum guaranteed return but also an upside capture the risky portfolio returns. His goals are thus conflicting since in order to maximize the upside capture he has to increase the total riskiness of the portfolio and this can result in a violation of the minimum return guarantee if the stock market experiences periods of declining returns or if the investment policy is not optimal. On the other side a low risk profile on the investment choices can assure the achievement of the minimum return guarantee, if properly designed, but leaves no opportunity for upside capture.

2 Minimum guaranteed return and constraints on the level of wealth

The relevance of the introduction of minimum guaranteed return products has grown in recent years due to financial market instability and to low level of interest rates on government (sovereign) and other bonds. This makes it more difficult to fix the level of the guarantee in order to attract potential investors. Moreover this may create potential financial instability and defaults due to the high levels of guarantees fixed in the past for contracts with long maturities, as the life insurance or pension fund contracts. See, for example, [8][20][30].

A range of guarantee features can be devised such as rate-of-return guarantee, among which the principal guarantee, i.e. with a zero rate of return, minimum benefit guarantee,

and real principal guarantee. Some of them are more interesting for participants in pension funds while other are more relevant for life insurance products or mutual funds. In the case of minimum return guarantee we ensure a deterministic positive rate of return (given the admissibility constraints for the attainable rate of returns); in the minimum benefit a minimum level of payments are guaranteed, at retirement date, for example. In the presence of nominal guarantee usually a fixed percentage of the initial wealth is guaranteed at a specified date while real or flexible guarantee are usually connected to an inflation index or to a capital market index.

The guarantee constraints can be chosen with respect to the value of terminal wealth or as a sequence of (possibly increasing) guaranteed returns. This choice may lead to the conditions of the financial products linked to the fund. The design of the guarantee is a crucial issue and has a consistent impact on the choice of the management strategies.

Not every value of minimum guarantee is reachable, no arbitrage arguments can be applied. The optimal design of a minimum guarantee has been considered and discussed in the context of pension fund management in [14]. Muermann et al. [25] analyzes the willingness of participants to a defined contribution pension fund to pay for a guarantee from the point of view of regret analysis.

Another issue which has to be tackled in the formulation is the fact that policies which give a minimum guaranteed return usually provide to policyholders also a certain amount of the return of the risky part of the portfolio invested in the equity market. This reduces the possibility of implementing a portfolio allocation based on Treasury bonds since no upside potential would be captured. The main objective is thus a proper combination of two conflicting goals, namely a guaranteed return, i.e. a low profile of risk, and at least part of the higher returns which could be granted by the equity market at the cost of a high exposure to the risk of not meeting the minimum return requirement.

A first possibility is to divide the investment decision into two steps. In the first the investor chooses the allocation strategy without taking care of the guarantee, while in the second step he applies a dynamic insurance strategy (see for example [15]).

Consiglio et al. [9] discuss a problem of asset and liability management for UK insurance products with guarantees. These products offer to the owners both a minimum guaranteed rate of return and the possibility to participate in the returns of the risky part of the portfolio invested in the equity market. The minimum guarantee is treated as a constraint and the fund manager maximizes the Certainty Equivalent Excess Return on Equity (CEExROE). This approach is flexible and allows one to deal also with the presence of bonuses and/or target terminal wealth.

Different contributions in the literature tackled the problem of optimal portfolio choices with the presence of a minimum guarantee both in continuous and discrete time also from the point of view of portfolio insurance strategies both for an European type guarantee and for an American type guarantee, see for example [10][11].

We consider the problem of formulating and solving an optimal allocation problem including minimum guarantee requirements and participation in the returns generated from the risky portfolio. These goals can be achieved both considering them as constraints or including them in the objective function. In the following we will analyze in more detail the second case in the context of dynamic tracking error problems which in our opinion provide

the more flexible framework.

3 Benchmark and tracking error issues

The introduction of benchmarks and of indexed products has greatly increased since the Capital Asset Pricing Model (see [27][23][24]) promoted a theoretical basis for index funds. The declaration of a benchmark is particularly relevant in the definition of the risk profile of the fund and in the evaluation of the performance of funds' managers. The analysis of the success in replicating a benchmark is conducted through tracking error measures.

Considering a given benchmark different sources of tracking error can be analyzed and discussed, see, for example [19]. The introduction of a liquidity component in the management of the portfolio, the choice of a partial replication strategy, the management expenses, among others, can lead to tracking errors in the replication of the behavior of the index designed as benchmark. This issue is particularly relevant in a pure passive strategy where the goal of the fund manager is to perfectly mime the result of the benchmark while it is less crucial if we consider active asset allocation strategies in which the objective is to create overperformance with respect to the benchmark. For instance, the choice of asymmetric tracking error measures allows us to optimize the portfolio composition in order to try to maximize the positive deviations from the benchmark. For the use of asymmetric tracking error measures in a static framework see, for example, [16][22][26].

For a discussion on risk management in presence of benchmarking, see Basak and Shapiro [4]. Alexander and Baptista [1] analyze the effect of a drawdown constraint, introduced to control the shortfall with respect to a benchmark, on the optimality of the portfolios in a static framework.

We are interested in considering dynamic tracking error problems with a stochastic benchmark. For a discussion on dynamic tracking error problems we refer to [2][5][7][13][17].

4 Formulation of the problem

We consider the asset allocation problem for a fund manager who aims at maximizing the return on a risky portfolio while preserving a minimum guaranteed return. Maximizing the upside capture increases the total risk of the portfolio, this can be balanced by the introduction of a second goal, i.e. the minimization of the shortfall with respect to the minimum guarantee level.

We model the first part of the objective function as the maximization of the over performance with respect to a given stochastic benchmark. The minimum guarantee itself can be modeled as a, possibly dynamic, benchmark. Thus the problem can be formalized as a double tracking error problem where we are interested in maximizing the positive deviations from the risky benchmark while in the same time we want to minimize the downside distance from the minimum guarantee. The choice of asymmetric tracking error measures allows us to properly combine the two goals.

To describe the uncertainty in the context of a multiperiod stochastic programming problem we use a scenario tree. A set of scenarios is a collection of paths from $t = 0$ to T , with probabilities π_{k_t} associated to each node k_t in the path: according to the information structure assumed this collection can be represented as a scenario tree where the current state corresponds to the root of the tree and each scenario is represented as a path from the origin to a leaf of the tree.

If we fix it as a minimal guaranteed return, without any requirement on the upside capture we obtain a problem which fits the portfolio insurance framework, see, for example, [3][6][18][21][28]. For portfolio insurance strategies there are strict restrictions on the choice of the benchmark which cannot exceed the return on the risk free security for no arbitrage conditions.

Let x_{k_t} be the value of the risky benchmark at time t in node k_t ; z_t be the value of the lower benchmark, the minimum guarantee, which can be assumed to be constant or with a deterministic dynamics, thus it does not depend on the node k_t . We denote with y_{k_t} the value of the managed portfolio at time t in node k_t . Moreover let $\phi_{k_t}(y_{k_t}, x_{k_t})$ be a proper tracking error measure which accounts for the distance between the managed portfolio and the risky benchmark, and $\psi_{k_t}(y_{k_t}, z_t)$ a distance measure between the risky portfolio and the minimum guarantee benchmark. The objective function can be written as

$$\max_{y_{k_t}} \sum_{t=0}^T \left[\alpha_t \sum_{k_t=K_{t-1}+1}^{K_t} \phi_{k_t}(y_{k_t}, x_{k_t}) - \beta_t \sum_{k_t=K_{t-1}+1}^{K_t} \psi_{k_t}(y_{k_t}, z_t) \right] \quad (1)$$

where α_t and β_t represent sequences of positive weights which can account both for the relative importance of the two goals in the objective function and for a time preference of the manager. For example, if we consider a pension fund portfolio management problem we can assume that the upside capture goal is preferable at the early stage of the investment horizon while a more conservative strategy can be adopted at the end of the investment period. A proper choice of ϕ_t and ψ_t allows us to define different tracking error problems.

The tracking error measures are indexed along the planning horizon in such a way that we can monitor the behavior of the portfolio at each trading date t . Other formulations are possible. For example, we can assume that the objective of a minimum guarantee is relevant only at the terminal stage where we require a minimum level of wealth z_T

$$\max_{y_{k_t}} \sum_{t=0}^T \left[\alpha_t \sum_{k_t=K_{t-1}+1}^{K_t} \phi_{k_t}(y_{k_t}, x_{k_t}) - \beta_T \sum_{k_T=K_{T-1}+1}^{K_T} \psi_{k_T}(y_{k_T}, z_T) \right]. \quad (2)$$

The proposed model can be considered a generalization of the tracking error model of Dembo and Rosen [12] who consider as an objective function a weighted average of positive and negative deviations from a benchmark, in our model we consider two different benchmarks and a dynamic tracking problem.

The model can be generalized in order to take into account a monitoring of the shortfall more frequent than the trading dates, see Dempster et al. [14].

We now present the formulation of the model in its arborescent form. We consider a manager who has to compose and manage his portfolio using n risky asset and a liquidity

component. We denote with $r_{k_t} = (r_{1\ k_t}, \dots, r_{n\ k_t})$ the vector of returns of the risky assets for the period $[t-1, t]$ in node k_t and with $r_{n+1\ k_t}$ the return on the liquidity component in node k_t . In order to account for transaction costs and liquidity component in the portfolio we introduce two vector of variables $a_{k_t} = (a_{1\ k_t}, \dots, a_{n\ k_t})$ and $v_{k_t} = (v_{1\ k_t}, \dots, v_{n\ k_t})$ denoting the value of each asset purchased and sold at time t in node k_t , while we denote with κ the proportional transaction costs.

Moreover we choose the mean absolute downside deviation tracking error measure both for the risky portfolio and for the guaranteed level of wealth in the objective function, that is we set

$$\phi_{k_t}(y_{k_t}, x_{k_t}) = [y_{k_t} - x_{k_t}]^+ = \theta_{k_t}^+; \quad (3)$$

$$\psi_{k_t}(y_{k_t}, z_t) = [y_{k_t} - z_t]^- = \gamma_{k_t}^-. \quad (4)$$

where $[y_{k_t} - x_{k_t}]^+ = \max[y_{k_t} - x_{k_t}, 0]$ and $[y_{k_t} - z_t]^- = \min[y_{k_t} - z_t, 0]$. The minimum guarantee can be assumed constant over the entire planning horizon or it can follow a deterministic dynamics, i.e it is not scenario dependent. Following [14] we assume that there is an annual guaranteed rate of return denoted with ρ . If the initial wealth is $W_0 = \sum_{i=1}^{n+1} x_{i0}$, then the value of the guarantee at the end of the planning horizon is $W_T = W_0(1 + \rho)^T$. At each intermediate date the value of the guarantee is given by $z_t = e^{\delta_{(t,T-t)}(T-t)} W_0(1 + \rho)^T$, where $e^{\delta_{(t,T-t)}(T-t)}$ is a discounting factor, i.e. the price at time t of a zcb which pays 1 at terminal time T .

The dynamic tracking error problem is

$$\max_{q_{k_t}, b_{k_t}, c_{k_t}} \sum_{t=0}^T \left[\alpha_t \sum_{k_t=K_{t-1}+1}^{K_t} \theta_{k_t}^+ - \beta_t \sum_{k_t=K_{t-1}+1}^{K_t} \gamma_{k_t}^- \right] \quad (5)$$

$$\theta_{k_t}^+ - \theta_{k_t}^- = y_{k_t} - x_{k_t} \quad (6)$$

$$\gamma_{k_t}^+ - \gamma_{k_t}^- = y_{k_t} - z_t \quad (7)$$

$$y_{k_t} = c_{k_t} + \sum_{i=1}^{n_1} q_{i k_t} + \sum_{j=1}^{n_2} b_{j k_t} \quad (8)$$

$$q_{i k_t} = (1 + r_{i k_t}) [q_{i b(k_t)} + a_{i b(k_t)} - v_{i b(k_t)}] \quad i = 1, \dots, n_1 \quad (9)$$

$$b_{j k_t} = (1 + r_{j k_t}) [b_{j b(k_t)} + a_{j b(k_t)} - v_{j b(k_t)}] \quad j = 1, \dots, n_2 \quad (10)$$

$$c_{k_t} = (1 + r_{c k_t}) [c_{b(k_t)} - \sum_{i=1}^{n_1} (\kappa^+) a_{i b(k_t)} + \sum_{i=1}^{n_1} (\kappa^-) v_{i b(k_t)} \quad (11)$$

$$+ \sum_{j=1}^{n_2} (\kappa^+) a_{j b(k_t)} + \sum_{j=1}^{n_2} (\kappa^-) v_{j b(k_t)} + \sum_{j=1}^{n_2} f_{k_t} b_{j b(k_t)}] \quad (12)$$

$$a_{i k_t} \geq 0 \quad v_{i k_t} \geq 0 \quad i = 1, \dots, n_1 \quad (13)$$

$$a_{j k_t} \geq 0 \quad v_{j k_t} \geq 0 \quad j = 1, \dots, n_2 \quad (14)$$

$$q_{i k_t} \geq 0 \quad i = 1, \dots, n_1 \quad (15)$$

$$b_{j k_t} \geq 0 \quad j = 1, \dots, n_2 \quad (16)$$

$$\theta_{k_t}^+ \geq 0 \quad \theta_{k_t}^- \geq 0 \quad (17)$$

$$\gamma_{k_t}^+ \geq 0 \quad \gamma_{k_t}^- \geq 0 \quad (18)$$

$$c_{k_t} \geq 0 \quad (19)$$

$$q_{i 0} = \bar{q}_i \quad i = 1, \dots, n_1 \quad (20)$$

$$b_{j 0} = \bar{b}_j \quad j = 1, \dots, n_2 \quad (21)$$

$$c_0 = \bar{c} \quad (22)$$

$$k_t = K_{t-1} + 1, \dots, K_t \quad (23)$$

$$t = 0, \dots, T \quad (24)$$

where we need to specify the value of the benchmark and the value of the minimum guarantee at each time and for each node. The stochastic benchmark y_{k_t} and the prices of the risky assets in the portfolio must be simulated according to given stochastic processes in order to build the corresponding scenario trees. Other dynamics for the minimum guaranteed level of wealth can be designed. In particular, we can discuss a time varying rate or return ρ_t along the planning horizon, or we can include the accrued bonuses as in [8].

A second approach to tackle the problem of the minimum return guarantee is to introduce probabilistic constraints in the dynamic optimization problem. Denoting with θ , the desired confidence level we can formulate the shortfall constraints both on the level of

wealth at an intermediate time t and on the terminal wealth as follow

$$Pr(W_t \leq z_t) \leq 1 - \theta \quad Pr(W_T \leq z_T) \leq 1 - \theta$$

where W_t is the random variable representing the level of wealth. Under the assumption of a discrete and finite number of realizations we can compute the shortfall probability using the values of the wealth in each node $W_{k_t} = \sum_{i=1}^{n+1} x_i k_t$. This gives origin to a chance constrained stochastic optimization problem which can be extremely difficult to solve due to non-convexities which may arise, see [14].

5 Concluding remarks

We discuss the issue of including in the formulation of a dynamic portfolio optimization problem both a minimum return guarantee and the maximization of the potential returns from a risky portfolio. To combine these two conflicting goals we formulate them in the framework of a double dynamic tracking error problem using asymmetric tracking measures.

References

- [1] Alexander, G.J., Baptista, A.M.: Portfolio selection with a drawdown constraint. *Journal of Banking and Finance* **30** (2006) 3171-3189
- [2] Barro, D., Canestrelli, E.: Tracking error: a multistage portfolio model. *Annals of Operations Research*, *forthcoming*.
- [3] Basak, S.: A general equilibrium of portfolio insurance. *Review of financial studies* **8** (1995) 1059-1090
- [4] Basak, S., Shapiro, A., and Tepla, L.: Risk management with benchmarking. Working Paper Series Asset Management SC-AM-03-16, Salomon Center for the study of financial institutions, NYU (2003)
- [5] Boyle, P., Tian, W.: Optimal portfolio with constraints. *Mathematical Finance* **17** (2007) 319-343
- [6] Brennan, M., Solanki, R.: Optimal portfolio insurance. *Journal of Financial and Quantitative Analysis* **16** (1981) 279-300
- [7] Browne, S.: Beating a moving target: optimal portfolio strategies for outperforming a stochastic benchmark. *Finance and Stochastics* **3** (1999) 255-271
- [8] Consiglio, A., Cocco, F., and Zenios S.A.: The Prometeia model for managing insurance policies with guarantees. In: Zenios, S.A. and Ziemba W.T. (Eds.) *Handbook of asset and liability management vol.2* 663-705, North-Holland (2007)

- [9] Consiglio, A., Saunders, D., and Zenios S.A.: Asset and liability management for insurance products with minimum guarantees: The UK case. *Journal of Banking and Finance* **30** (2006) 645-667
- [10] Deelstra, G., Grasselli, M., Koehl, P-F.: Optimal investment strategies in the presence of a minimum guarantee. *Insurance: Mathematics and Economics* **33** (2003) 189-207
- [11] Deelstra, G., Grasselli, M., Koehl, P-F.: Optimal design of the guarantee for defined contribution funds. *Journal of Economic Dynamics and Control* **28** (2004) 2239-2260
- [12] Dembo, R., Rosen, D.: The practice of portfolio replication, a practical overview of forward and inverse problems. *Annals of Operations Research* **85** 267-284
- [13] Dempster, M.H.A., Thompson, G.W.P.: Dynamic portfolio replication using stochastic programming. In M.H.A. Dempster (Ed.), *Risk Management: value at risk and beyond* (pp. 100-128). Cambridge University Press.
- [14] Dempster, M.A.H., Germano, M., Medova, E.A., Rietbergen, M.I., Sandrini, F., and Scowston, M.: Designing minimum guaranteed return funds. *Research Papers in Management Studies*, University of Cambridge, WP 17/2004 (2004)
- [15] El Karoui N., Jeanblanc, M., Lacoste, V.: Optimal portfolio management with American capital guarantee. *Journal of Economic Dynamics and Control* **29** (2005) 449-468
- [16] Franks, E.C.: Targeting excess-of-benchmark returns. *The Journal of Portfolio Management* **18** (1992) 6-12
- [17] Gaivoronski, A., Krylov, S., van der Vijst, N.: Optimal portfolio selection and dynamic benchmark tracking. *European Journal of Operational Research* **163** 115-131
- [18] Grossman, S.J., Zhou, Z.: Equilibrium analysis of portfolio insurance. *Journal of Finance* **51** (1996) 1379-1403
- [19] Halpern, P., Kirzner, E.: Quality standards for index products. working paper Rotman School of Management, University of Toronto (2001)
- [20] Hochreiter, R., Pflug, G., Paulsen V.: Design and management of Unit-linked Life insurance Contracts with guarantees. In: Zenios, S.A. and Ziemba W.T. (Eds.) *Handbook of asset and liability management vol.2* 627-662, North-Holland (2007)
- [21] Jensen, B.A., Sorensen, C.: Paying for minimal interest rate guarantees: Who should compensate whom? *European Financial Management* **7** (2001) 183-211
- [22] Konno, H., Yamazaki, H.: Mean absolute deviation portfolio optimization model and its applications to Tokyo stock market. *Management Science* **37** (1991) 519-531
- [23] Lintner, J.: The Valuation of Risky Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budget. *Review of Economics and Statistics* **47** (1965) 13-37

- [24] Mossin, J.: Equilibrium in a Capital Asset Market. *Econometrica* **34** (1966) 768-783
- [25] Muermann, A., Mitchell, O.S., Volkman, J.M.: Regret, portfolio choice, and guarantees in defined contribution schemes. *Insurance: Mathematics and Economics* **39** (2006) 219-229
- [26] Rudolf, M., Wolter, H-J., Zimmermann, H.: A linear model for tracking error minimization. *Journal of Banking and Finance* **23** (1999) 85-103
- [27] Sharpe, W.F.: Capital Asset Prices: a Theory of Market Equilibrium Under Conditions of Risk. *Journal of Finance* **19(3)** (1964) 425-442
- [28] Sorensen, C.: Dynamic asset allocation and fixed income management. *Journal of Financial and Quantitative Analysis* **34** (1999) 513-531
- [29] Zenios, S.A. and Ziemba W.T. (Eds.): Handbook of asset and liability management vol.2. North-Holland (2007)
- [30] Ziemba, W.T.: The Russel-Yasuda Kasai, InnoALM and related models for pensions, Insurance companies and high net worth individuals. In: Zenios, S.A. and Ziemba W.T. (Eds.) Handbook of asset and liability management vol.2 861-962, North-Holland (2007)